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1981 J. Phys. A: Math. Gen. 14 L519

(http://iopscience.iop.org/0305-4470/14/12/008)

J. Phys. A: Math. Gen. **14** (1981) L519–L525. Printed in Great Britain

LETTER TO THE EDITOR

The Gross–Neveu model: dynamical symmetry breaking at a Kosterlitz–Thouless phase transition

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Received 1 October 1981

Abstract. The Gross–Neveu model can be interpreted as the massive, scaling limit theory derived from a model which undergoes a Kosterlitz–Thouless phase transition.

1. Introduction

The Gross–Neveu model (Gross and Neveu 1974) is a $(1+1)$-dimensional field theory of interacting fermions, with Lagrangian density

$$L_{\mathrm{GN}} = \sum_{i=1}^{N} \bar{\psi}_i (\mathrm{i}\gamma \cdot \boldsymbol{\partial}) \psi_i + \frac{1}{2} g^2 \left[\sum_{i=1}^{N} \bar{\psi}_i \psi_i \right]^2 \tag{1.1}$$

where g^2 is a dimensionless coupling constant. This model is asymptotically free and exhibits dynamical symmetry breaking: although the fermions start out massless, they become massive due to the composite field $\Sigma_{i=1}^{N} \bar{\psi}_i \psi_i$ acquiring a non-vanishing vacuum expectation value. This is an example of dimensional transmutation: the single parameter of the model, g^2, is replaced by a single massive parameter, which can be taken to be the fermion mass.

In this Letter I show that this phenomenon can be understood as originating from a Kosterlitz–Thouless phase transition (Kosterlitz and Thouless 1973) which occurs in a related two-parameter model which includes the Gross–Neveu model as a very special case. The Gross–Neveu model is the scaling limit of this other model.

It has been known for some time that a one-parameter interacting field theory can be obtained from a two-parameter theory which undergoes a second-order phase transition. Suppose the two parameters are T and μ, where T is dimensionless and μ has dimensions of mass. Near a critical point T_c, the physical masses behave like $\mu|T-T_c|^{\nu}$. If μ is taken to infinity as T goes to T_c in such a way that the physical masses stay fixed, a new theory, the scaling limit of the old theory, is obtained. For a lattice model, μ^{-1} is the lattice spacing, and the new theory obtained is the continuum limit theory.

In a Kosterlitz–Thouless phase transition, physical masses vanish much more quickly as T goes to T_c. Typically (Kosterlitz 1974) the physical masses vanish as

$$\exp(-a/|T-T_c|^{\nu'}). \tag{1.2}$$

† Supported by the National Science Foundation.

Obviously, it should be possible to construct a scaling limit theory from a theory which has a Kosterlitz–Thouless phase transition. My claim is that the Gross–Neveu model can be so obtained. The principal tools required to demonstrate this are the Bose form of the Gross–Neveu model (Witten 1978) and the analysis of the sine-Gordon model's critical behaviour by Amit *et al* (1980).

This result is consistent with the suggestion of McCoy and Wu (1979) that a novel field theory can be constructed using the infinite-order phase transition in the XYZ spin chain, which is equivalent to a lattice version of the Thirring model. The massive Thirring model is equivalent to the sine-Gordon model, which is the prototypical example of a field theory with a Kosterlitz–Thouless phase transition. Witten (1978) has shown that for $N=2$, the Gross–Neveu model is equivalent to two decoupled sine-Gordon models.

2. Bose form of the Gross–Neveu model

The Bose form of the Gross–Neveu model has been discussed by Witten (1978). An interesting discussion of the group-theoretical aspects of this equivalence has been given by Shankar (1981). The fermion fields are replaced by N scalar fields $\varphi_1, \ldots, \varphi_N$. The Bose form of the Lagrangian is

$$L_{\mathrm{B}} = \tfrac{1}{2}(1+g^2/2\pi)\sum_i (\partial_\mu\varphi_i)^2 + (g^2m^2/2\pi^2)\sum_{i\neq j} N_m \cos(4\pi)^{1/2}\varphi_i N_m \cos(4\pi)^{1/2}\varphi_j. \tag{2.1}$$

The symbol N_m indicates normal-ordering with respect to the arbitrary mass m. Although L_{B} appears to depend on m, it does not, because

$$mN_m \cos(4\pi)^{1/2}\varphi_i = \mu N_\mu \cos(4\pi)^{1/2}\varphi_i. \tag{2.2}$$

It might be objected that the Gross–Neveu model is renormalisable, not super-renormalisable, and the normal-ordering in (2.1) is inadequate to render the theory finite. This is true, but misses the point. The coupling constant g^2 is the bare coupling constant. Order by order, in an expansion about massless free-field theory, L_{GN} and L_{B} will give the same (bare) Green functions, which must be renormalised to be free of divergences.

The factor $(1+g^2/2\pi)$ multiplying the kinetic part of L_{B} is inconvenient; it would be nice to put it somewhere else. For a classical theory, this can be accomplished by a rescaling of the fields. For a quantum field theory this is problematic. In this case, the normal ordering of the cosine operators would not match the actual divergence of the operator. Therefore, it is best to remove the normal ordering, and then rescale the fields. Then L_{B} is given by

$$L_{\mathrm{B}} = \sum_i \tfrac{1}{2}(\partial_\mu\varphi_i)^2 + \sum_{i\neq j} \alpha_0\Lambda^2 \cos\beta_0\varphi_i \cos\beta_0\varphi_j. \tag{2.3}$$

The cut-off Λ replaces m, while α_0 and β_0 are functions of g:

$$\alpha_0 = g^2/2\pi^2 \tag{2.4}$$

$$\beta_0^2 = 4\pi/(1+g^2/2\pi). \tag{2.5}$$

The origin of the phase transition is the marginality of the interaction part of the Lagrangian when $\beta_0^2 = 4\pi$. The operator $\cos\beta_0\varphi_i$ has dimension zero in units of mass, but its anomalous dimension is $\beta_0^2/4\pi$. If $\beta_0^2 < 4\pi$, the product of two cosine operators will be a relevant operator in the renormalisation group sense, and L_B will be super-renormalisable. If $\beta_0^2 = 4\pi$, the anomalous dimension of this operator is two, so it is marginal, and L_B is merely renormalisable. The Lagrangian L_B is non-renormalisable if $\beta_0^2 > 4\pi$. Because the Gross–Neveu model is asymptotically free, the base coupling constant g goes to zero as the cut-off is removed, so β_0^2 goes to the critical value, 4π, at the same time.

It is easy to interpret this model as an equivalent problem in classical statistical mechanics. The generating functional Z of Euclidean Green functions can be written as a functional integral, and expanded as a power series in α_0. At each order in α_0, the functional integral can be carried out, which recasts Z into the form of a grand partition function of classical particles interacting via two-body potentials. In the simpler case of the sine-Gordon model, this equivalence yields a simple Coulomb gas of positive and negative charges. For this model, a Coulomb gas is again obtained, but with N different kinds of charges. Each charge interacts only with charges of its own kind, but each particle carries two different charges. The origin of the phase transition is the collapse of the particles into tightly bound neutral pairs as described by Kosterlitz and Thouless (1973).

3. Perturbation theory

In order to study the critical behaviour of the model defined by equation (2.8), I will employ the techniques of Amit *et al* (1980). In this section I calculate the one-particle-irreducible two-point function $\Gamma^{(2)}$ to second order in perturbation theory about the multicritical point $\alpha_0 = 0$, $\beta_0^2 = 4\pi$. The two-point function is a diagonal $N \times N$ matrix, and all non-zero entries are equal. This is a consequence of the symmetry

$$\begin{aligned} &\varphi_i \to -\varphi_i \\ &\varphi_j \to +\varphi_j \qquad (j \neq 1). \end{aligned} \tag{3.1}$$

The details of the calculation are very similar to those for the sine-Gordon model.

It is convenient to work from now on in Euclidean space. The Euclidean Lagrangian density L_E is given by

$$L_E = \sum_i \tfrac{1}{2}(\nabla\varphi_i)^2 - \frac{\alpha_0}{a^2}\sum_{i \neq j} \cos \beta_0\varphi_i \cos \beta_0\varphi_j \tag{3.2}$$

where $a = \Lambda^{-1}$. The graphical rules, shown in figure 1, generalise those of Kosterlitz (1975).

The $O(\alpha_0)$ contribution to $\Gamma^{(2)}$ is the first graph shown in figure 1; the index j must be summed from 1 to N, with i excluded. The five $O(\alpha_0^2)$ contributions are shown in figure 2; all indices except i must be summed from 1 to N. Graph (c) is ambiguous as drawn, but a particle of type i is both entering and emerging. It should be noted that there are graphs which could be included in (a), (b), or (c), and likewise (d) or (e); it is important not to overcount.

As usual, $\Gamma^{(2)}$ can be written as

$$\Gamma^{(2)}(p^2) = p^2 + \Sigma(p^2) \tag{3.3}$$

where Σ is the one-particle-irreducible self-energy. As in Kosterlitz (1974), it is necessary to add extra mass terms to L_E in order to regulate the infrared divergences of

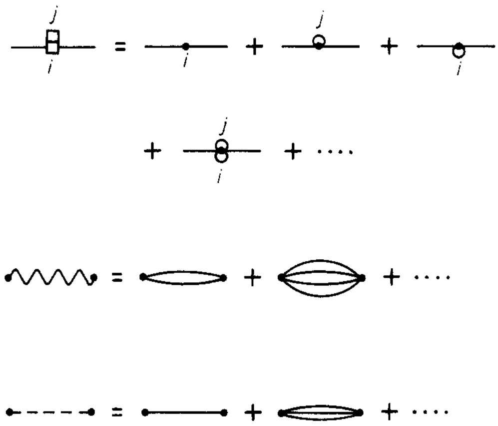

Figure 1. Basic graphical rules.

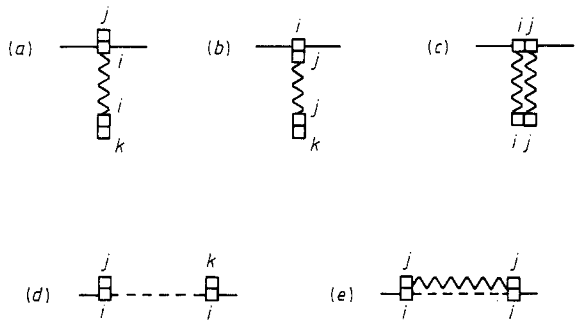

Figure 2. $O(\alpha^2)$ contributions to $\Gamma^{(2)}$.

the expansion:

$$L_E \to L_E + \sum_i \tfrac{1}{2} m^2 \varphi_i^2. \tag{3.4}$$

This will not affect the renormalisation group equations for β and α.

The $O(\alpha_0)$ contribution to Σ, $\Sigma^{(1)}$, is

$$\Sigma^{(1)} = (\alpha_0 \beta_0^2 J/a^2) 2(N-1) \tag{3.5}$$

where

$$J \equiv \exp[-\beta_0^2 \Delta(x=0)]. \tag{3.6}$$

and $\Delta(x)$ is the Euclidean propagator.

In order to write down a simple form for the $O(\alpha_0^2)$ contributions to Σ, it is convenient to define

$$C(x) \equiv \cosh \beta_0^2 \Delta(x) \tag{3.7a}$$

$$S(x) \equiv \sinh \beta_0^2 \Delta(x). \tag{3.7b}$$

The contributions of the five diagrams in figure 2 are given by

$$\Sigma^{(2)} = (\alpha_0 J/a^2)^2\beta_0^2 \int d^2x\, \{8(N-1)^2[C(x)-1]+4(N-1)[C(x)-1]^2$$
$$\times \exp(ip\cdot x)4(N-1)^2[S(x)-\beta_0^2\Delta(x)]$$
$$-\exp(ip\cdot x)4(N-1)S(x)[C(x)-1]\}. \tag{3.8}$$

The critical value of β_0^2 is determined by $\Sigma^{(1)}$. The divergences can be regulated by the replacement of x^2 by x^2+a^2. The short distance behaviour ($m|x|\ll 1$) of $\Delta(x)$ is

$$\Delta(x)\simeq(-1/4\pi)\ln[m^2|x|^2] \tag{3.9}$$

so

$$\Sigma^{(1)} = 2(N-1)(\alpha_0\beta_0^2/a^2)\exp[(\beta_0^2/4\pi)\ln(m^2a^2)]. \tag{3.10}$$

The critical value of β_0^2 is 4π. I define

$$\delta_0 = (\beta_0^2/4\pi)-1 \tag{3.11}$$

for use as an expansion parameter.

It is necessary to find all divergences of $\Gamma^{(2)}$ to second-order in a double expansion in α_0 and δ_0. This is easy for $\Sigma^{(1)}$:

$$\Sigma^{(1)}\simeq\alpha_0 8\pi(N-1)(1+\delta_0)m^2[1+\delta_0\ln(m^2a^2)]. \tag{3.12}$$

As in the sine-Gordon model, the divergences in $\Sigma^{(2)}$ can be catalogued by examining only $\Sigma^{(2)}(p^2=0)$ and $\partial\Sigma^{(2)}/\partial p^2|_{p^2=0}$. The divergent part of $\Sigma^{(2)}(p^2=0)$ is

$$\Sigma^{(2)}(p^2=0) = -\alpha_0^2 m^2 8\pi^2(N-1)(N-2)\ln(m^2a^2) \tag{3.13}$$

while the other divergence is given by

$$\partial\Sigma^{(2)}/\partial p^2|_{p^2=0} = -\alpha_0^2 8\pi^2(N-1)\ln(m^2a^2). \tag{3.14}$$

Combining these results, I find $\Gamma^{(2)}$ is given by

$$\Gamma^{(2)}(p^2) = p^2+m^2+8\pi(N-1)\alpha_0 m^2[1+\delta_0\ln(m^2a^2)]$$
$$-8\pi^2(N-1)(N-2)\alpha_0^2m^2\ln(m^2a^2)-\pi^2\alpha_0^2(N-1)p^2\ln(m^2a^2) \tag{3.15}$$

up to finite parts.

4. The renormalisation group

Two renormalisation constants, Z_α and Z_φ, suffice to remove all divergences in $\Gamma^{(2)}$. Renormalised quantities are defined by

$$\Gamma_R^{(N)} = Z_\varphi^{N/2}\Gamma^{(N)} \tag{4.1}$$

$$\alpha_0 = Z_\alpha\alpha \tag{4.2}$$

$$\beta_0^2 = Z_\varphi^{-1}\beta^2 \tag{4.3}$$

$$m_0^2 = Z_\varphi^{-1}m^2. \tag{4.4}$$

The only unusual feature is the renormalisation of β_0^2; see Kosterlitz (1974) for a discussion of this point.

The requirement that $\Gamma_R^{(2)}$ be finite fixes Z_α and Z_φ. To second order they are given by

$$Z_\alpha = 1 - \delta \ln(m^2a^2) + \pi(N-2)\alpha^2 \ln(m^2a^2) \tag{4.5}$$

$$Z_\varphi = 1 + \pi^2\alpha^2(N-1) \ln(m^2a^2). \tag{4.6}$$

The renormalisation group β functions are defined in the usual way:

$$\begin{aligned}\beta_\alpha &= -\alpha a(\partial \ln Z_\alpha/\partial a)\\ &= 2\delta\alpha - 2\pi(N-2)\alpha^2\end{aligned} \tag{4.7}$$

$$\begin{aligned}\beta_\delta &= a(\partial\delta/\partial a) = (1+\delta)a(\partial \ln Z_\varphi/\partial a)\\ &= 2\pi^2(N-1)\alpha^2.\end{aligned} \tag{4.8}$$

The renormalisation group flow is shown in figure 3. It has the characteristic hyperbolic shape associated with Kosterlitz–Thouless phase transitions. There are three regions. Region I describes the behaviour of the model when $\beta_0^2 < 4\pi$: this phase is massive and asymptotically free. In region III the model is non-renormalisable. The infrared flow of the renormalisation group trajectories leads into a line of infrared-stable fixed points. This phase is analogous to the low-temperature spin-wave phase of the XY model.

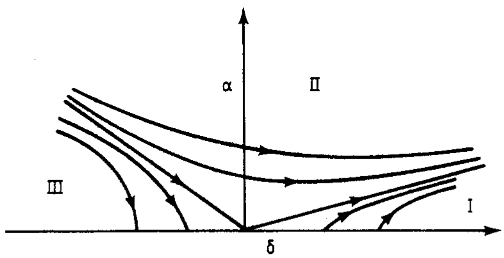

Figure 3. Renormalisation group flow.

Note that the flow is not symmetric about $\delta = 0$, except in the case $N = 2$. This occurs because L_B is not invariant under $\alpha \to -\alpha$, unlike the sine-Gordon case. When $N = 2$, the flow equations (4.7) and (4.8) reduce to the symmetric sine-Gordon form because the $N = 2$ Gross–Neveu model is equivalent to two decoupled sine-Gordon models (Witten 1978).

The separatrices of the flow diagram can be obtained by substituting $\alpha = m\delta$ into equations (4.7) and (4.8) and solving for m. The result is

$$m_1 = -1/\pi \tag{4.9}$$

and

$$m_+ = 1/\pi(N-1). \tag{4.10}$$

From equations (2.4) and (2.5), it can be seen that for the Gross–Neveu model, α and δ are given to lowest order by

$$\alpha = g^2/2\pi^2 \tag{4.11}$$

$$\delta = -g^2/2\pi. \tag{4.12}$$

Thus the Gross–Neveu model is described by the renormalisation group flow on the left-hand separatrix. In fact, substitution of (4.11) and (4.12) into (4.7) and (4.8) gives

$$\beta_g = -(N-1)g^3/2\pi \tag{4.14}$$

which is the lowest-order Gross–Neveu model beta function. Thus we see that the Gross–Neveu model and its dynamical symmetry breaking are associated with a Kosterlitz–Thouless phase transition. It is easy to show that there is a phase transition as the left-hand separatrix is approached from the left-hand side. This proves the claim made at the beginning of this Letter.

After this work was completed, I received a preprint from Girardello *et al* (1981) which studies the $N=2$ Gross–Neveu model using its equivalence to two decoupled sine-Gordon models. Their conclusions are similar to mine, but are restricted to the case $N=2$.

References

Amit D J, Goldschmidt Y Y and Grinstein G 1980 *J. Phys. A: Math. Gen.* **13** 585
Girardello L, Immirzi G and Rossi P 1981 *MIT preprint* CTP #934
Gross D J and Neveu A 1974 *Phys. Rev* **D10** 3235
Kosterlitz J M 1974 *J. Phys. C: Solid State Phys.* **7** 1046
Kosterlitz J M and Thouless D J 1973 *J. Phys. C: Solid State Phys.* **6** 1181
McCoy B M and Wu T T 1979 *Phys. Lett.* **87B** 50
Shankar R 1981 *Yale preprint YTP* #81-06
Witten E 1978 *Nucl. Phys. B* **142** 285